and from (60) the well-known formula is derived (see Ref. 2):

$$I(0, \eta) = \frac{\varphi(\eta)}{\sqrt{1 - \lambda}} \tag{62}$$

Now let the scattering indicatrix have the form

$$x(\gamma) = 1 + x_1 \cos\gamma \tag{63}$$

For such a case

$$A(\eta) = 1 - \frac{1}{\eta} \varphi_1^0(\eta) \tag{64}$$

and from (60) it follows that

$$I(0, \eta) = \frac{1}{\eta} \cdot \frac{\varphi_1^0(\eta)}{1 - \lambda} \tag{65}$$

3) Let us now determine the angular distribution of the intensity of radiation $I(\eta)$ diffusely transmitted by the medium which has a large optical depth. In such a case, it should be considered that the radiational sources are located at an infinitely large depth.

Let us first derive the equations for the emission probability of a quantum from a very large optical depth ($\tau \gg 1$). Physical meaning implies that this probability will be dependent only on τ and η and, in addition, that the expression for $P(\tau, \eta)$ must be sought in the following form:

$$P(\tau, \eta) = C(\eta) \cdot e^{-k\tau} \tag{66}$$

where k is an unknown constant and $C(\eta)$ is an unknown function.

Substitution of (66) in (48) gives

$$-kC(\eta) = -\frac{1}{\eta} C(\eta) + 2\pi \int_0^1 C(\eta')p_0(0, \eta', \eta) \frac{d\eta'}{\eta'} \tag{67}$$

Equations (67) and (32) indicate that

$$C(\eta) = \frac{\lambda}{2} \eta \sum_{i=0}^{n} \frac{c_i x_i \varphi_i^0(\eta)}{1 - k\eta} \tag{68}$$

where

$$c_i = \int_0^1 C(\eta)P_i(\eta) \frac{d\eta}{\eta} \tag{69}$$

Equation (68) determines function $C(\eta)$. Constants c_i included in (68) are derived from the system of linear algebraic equations

$$c_j = \frac{\lambda}{2} \sum_{i=0}^{n} x_i c_i a_{ij} \tag{70}$$

where

$$a_{ij} = \int_0^1 \frac{\varphi_i^0(\eta)P_j(\eta)}{1 - k\eta} d\eta \tag{71}$$

System (70) is derived by multiplying (68) by $P_j(\eta) \cdot (1/\eta)$ and integrating over η from 0 to 1. The condition of nontrivial solution of this system determines the value of constant k. It should be pointed out that this method determines $C(\eta)$ only to within a constant factor.

Taking (66), (68), (33), and (47) into account for the angular distribution of the intensity of diffusely transmitted radiation we have

$$I(\eta) = \frac{\lambda}{2} \sum_{i=0}^{n} \frac{x_i c_i \varphi_i^0(\eta)}{1 - k\eta} \tag{72}$$

The investigated problem was previously solved by Ambartsumian[7] by the method of combined layers which he developed. Solution by the probabilistic method described here leads to the goal much faster and therefore may be of some interest.

The present paper solves the problem of determining the intensity of radiation emitted from a semi-infinite medium with an arbitrary scattering indicatrix for any distribution of sources, the power of which, in a general case, can be dependent on depth and direction. The solution is obtained by using the probabilistic concept of emission of a quantum from the medium for a given direction.

It is demonstrated by Sobolev[2, 3] that on the basis of the emission probability of a quantum from the medium it is possible to determine also the radiation field within the medium. The luminous mechanism within the medium will be determined in Part II. Application of the Laplace transforms to the problems under investigation will be demonstrated there also.

Summary

The problem of diffusion of radiation in a semi-infinite medium with nonspherical indicatrix of scattering is considered with the aid of the probabilistic method. In Sec. 1 the probability of emergence for a photon from a medium in a given direction is introduced. In Sec. 2 the equations for the probability of emergence for a photon from a medium are given. In Sec. 3 the intensity of radiation emerging from a medium illuminated by parallel rays is derived. The expressions for the intensity of radiation emerging from a medium for different sources of radiation are derived in Sec. 4.

References

1 Sobolev, V. V., Astron. Zh. (Astron. J.) **28,** no. 5 (1951).

2 Sobolev, V. V., *Transmission of Radiant Energy in Stellar and Planetary Atmospheres* (Gostekhizdat, Moscow, 1956).

3 Sobolev, V. V., Doklady Akad. Nauk SSSR (Proc. Acad. Sci. USSR) **116,** no. 1 (1957); **120,** no. 1 (1958).

4 Ambartsumian, V. A., Zh. Eksperim. i Teor. Fiz. (Soviet Physics—JETP) **13,** no. 9, 10 (1943).

5 Sobolev, V. V., Doklady Akad. Nauk SSSR (Proc. Acad. Sci. USSR) **69,** no. 3 (1949).

6 Ueno, S., J. Math. and Mech. **7,** no. 4 (1958).

7 Ambartsumian, V. A., Doklady Acad. Nauk SSSR (Proc. Acad. Sci. USSR) **43,** no. 3 (1944).

Reviewer's Comment

In this paper, the author extends Sobolev's method of the solution of radiative transfer problem in semi-infinite medium for anisotropic scattering. This solution is based on the probabilistic method, whereas Chandrasekhar[1] indicated the solution of the same problem based on the use of the equation of radiative transfer and the principles of invariance. The equivalence of these two different methods has been demonstrated for the case of isotropic scattering by Ueno.[2] Chandrasekhar's method is also discussed in an exact mathematical analysis by Busbridge.[3]

The problem, investigated by the author in the present paper, can be considered a special case of the problem of radiative transfer in a plane-parallel medium of finite optical depth, studied for anisotropic scattering by Churchill et al.[4] and by Yanovitski.[5] The latter author indicated the solution of an inhomogeneous medium, in which the albedo for single scattering, as well as the indicatrix of scattering (phase function), vary with optical depth. Yanovitski uses the auxiliary equation (for the source function) as the starting point for his discussion, whereas Churchill et al. extend Chandrasekhar's solution for the case of variable albedo for single scattering. By a proper limiting process, in which the optical thickness of the medium increases over all limits, the solution for finite medium reduces to that derived by the author.

The scattering process in the radiative transfer problem, as formulated by the author and by the forementioned investigators, is defined only for the intensity of the scattered radiation and is given by a simple scalar function. However, for the cases of anisotropic scattering which occur in the planetary and stellar atmospheres, the scattering produces polarization, which substantially affects the intensity distribution, as demonstrated by Chandrasekhar. A more realistic solution of radiative transfer in a medium with anisotropic scattering requires the introduction of polarization effects, neglected in the forementioned studies. As shown by Chandrasekhar, the simple scalar equations have to be replaced by vectorial equations, the intensity or quantum emission by a matrix with four elements, the so-called Stokes parameters. In this case, it is difficult to use the probabilistic method, and it is preferable to solve the problem with the use of either the principles of invariance or the method of invariant imbedding (Bellman and Kalaba[6]). The resulting equations for the matrices defining the diffuse reflection and transmission have been derived recently by Sekera[7] and the method of their solution demonstrated for the case of Rayleigh scattering in a finite plane-parallel inhomogeneous medium.

—Z. Sekera
Professor of Meteorology
University of California

[1] Chandrasekhar, S., *Radiative Transfer* (Oxford University Press, New York, 1950).
[2] Ueno, S., Astrophys. J. **132**, 729 (1960).
[3] Busbridge, I. W., "Mathematics of radiative transfer," *Cambridge Tracts* (Cambridge University Press, New York, 1960), No. 50.
[4] Churchill, S. W., Chu, Ch. M., Evans, L. B., Tien, L., and Pang, S., Univ. of Michigan, Ann Arbor, Annual Rept. DASA-1257, 03675-I-F (1961).
[5] Yanovitski, E. G., Astron. Zh. (Astron. J.) **38**, 912 (1961), or Soviet Astronomy-AJ **5**, 697 (1962).
[6] Bellman, R. and Kalaba, R., Proc. Natl. Acad. Sci. **42**, 629 (1956).
[7] Sekera, Z., Proc. Interdisc. Conference on Electromagnetic Scattering, Potsdam, N. Y., and in Astrophys. J. (to appear).

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Theory of an Inertial System

V. A. Karakashev

In this article, the possibility of obtaining a geographically oriented accelerometer platform with a natural oscillation period of 84.4 min is examined in the first approximation. The influence of gyroscope drift on the motion of a gyrostabilized platform and an accelerometer platform is studied, and errors in the determination of the coordinates of a moving object generated by drift of the "reference" coordinate system are derived. The study is conducted under the assumption that ideal elements are used in the system (with the exception of the floating gyroscopes), i.e., elements not containing inherent instrumental and design errors.

Independent of type and construction, the use of any inertial system is based on the determination of a course which is applied to automatic or semi-automatic control of a moving object. The self-contained determination of the instantaneous location and other elements of the motion of an object is accomplished in inertial systems by measuring acceleration, gravitation, angular velocity, and time in some previously chosen coordinate system.

1. The inertial system under examination has the following layout (Fig. 1).

A platform with two orthogonally placed accelerometers A_x and A_y for measuring acceleration along the x and y axes was installed on a three-axis gyrostabilized platform, which maintained constant orientation relative to the fixed stars, using an "equatorial support," the axis of which was directed along the axis of the earth's diurnal rotation[3] (Fig. 1).

The accelerometer platform was maintained in a geographic coordinate system by servosystems which included various computers, a clock, servomotors, and other elements.

Coriolis and centripetal accelerations are eliminated from the output signals of the accelerometers by special computers; the corrected output signals of the accelerometers go to the inputs of corresponding integrators.[2]

Let us examine the operation of the servocircuits. When an object's latitude φ undergoes accelerated change, a signal from accelerometer A_y enters the integrator I_{Iy}, the output signal of which is proportional to the object's velocity V_1. The correcting device CD_y converts the output signal of integrator I_{Iy} into a signal which is proportional to the angular velocity V_1/R, which is again integrated by the integrator I_{IIy} and applied to the amplifier U_y. The amplified signals enter the control winding of the servomotor SM_x, and the latter turns the accelerometer about the axis of suspension x.

When the object's longitude λ undergoes accelerated change, a signal from accelerometer A_x enters integrator I_{Ix}. The output signal of the correcting device CD_x, proportional to $V_2/R \cos\varphi$, enters the integrator I_{IIx}, is again integrated, and enters the summation device SD. Here it is added to a signal from the clock C, proportional to the angle of turning due to the earth's diurnal rotation. After amplification in the amplifier U_x, the signal u_{SD} enters the control winding of the servomotor SM_{zG}, which turns the accelerometer platform about the O_1z_G axis.

The sensitive elements of the gyrostabilized platform are three floating inertial gyroscopes (FIG) whose angular momenta are oriented perpendicular to the axes of stabilization. The output signals of the gyroscopes are distributed through a

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